

Chiellini integrability condition, planar isochronous systems and Hamiltonian structures of Liénard equation

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Abstract

Using a novel transformation involving the Jacobi Last Multiplier (JLM) we derive an old integrability criterion due to Chiellini for the Liénard equation. By combining the Chiellini condition for integrability and Jacobi's Last Multiplier the Lagrangian and Hamiltonian of the Liénard equation is derived. We also show that the Kukles equation is the only equation in the Liénard family which satisfies both the Chiellini integrability and the Sabatini criterion for isochronicity conditions. In addition we examine this result by mapping the Liénard equation to a harmonic oscillator equation using tacitly Chiellini's condition. Finally we provide a metriplectic and complex Hamiltonian formulation of the Liénard equation through the use of Chiellini condition for integrability.

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1 Introduction

A second-order nonlinear differential equation of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$ where f and g are arbitrary real valued functions defined in an interval $I \subset \mathbb{R}$ is usually referred to as a Liénard equation. There exists a vast literature on this equation owing to its widespread applications in several branches of the applied sciences. Equations of the Liénard type are interesting because they display several novel features such as the existence of limit cycles, isochronicity, Hamiltonian structures etc.

A fundamental problem, common to any ordinary differential equation (ODE), is that of finding its solutions whenever they exist. As early as 1931 Cheillini [4] in course of his investigations on the integrability of second-order ODEs of the Liénard type arrived at the following condition for their integrability, namely

$$\frac{d}{dx} \left(\frac{g}{f} \right) = sf \quad (1.1)$$

where s is a constant. This condition is today referred to as the Cheillini condition and as has been recently proved to be extremely useful in the construction of exact solutions after transforming the Liénard equation to a first-order Abel equation of the first kind [8, 13, 14, 6, 19, 20].

On the other hand the Jacobi Last Multiplier (JLM), which was introduced in the context of a system of first-order ODEs more than 150 years ago by Carl Jacobi [?], is also concerned with the possibility of reducing a system of first-order ODEs to a quadrature and is therefore essentially related to the issue of integrability. There is however another interesting application of the JLM specially for second-order ODEs. It may be shown that if the JLM is known for a second-order ODE (or equivalently for a first-order planar differential system) then it is possible to deduce a Lagrangian for the system under consideration [9, 24, 25, 26]. For the Liénard equation it has been observed by the authors that the problem of finding a JLM and thereafter a Lagrangian for the equation is intimately related to the satisfaction of Cheillini's condition of integrability. This observation open up the door for investigation of the Hamiltonian structures of second-order ODEs of the Liénard type. A very promising way to algebrize the dynamics of a dissipative system is the metriplectic framework to which we shall be coming shortly. Recently much research [7, 11, 28] has been devoted to the identification of isochronous dynamical systems whose motions are completely periodic in their phase space. A center of a planar dynamical system is said to be isochronous if all cycles near it have same period. For an equation of the Liénard type the problem of determining the conditions on the functions f and g such that the center is isochronous was solved by Sabatini [30, 5] who enunciated the requisite conditions to be satisfied by f and g . Closely related to this problem is the question of finding a suitable transformation for mapping a planar dynamical system to that of a linear harmonic oscillator which is known to be isochronous. As the latter is integrable it is natural to enquire if in general the Cheillini condition, which also imposes conditions on f and g , and the conditions necessary for isochronicity as deduced by Sabatini are compatible or not.

It is obvious that formally the Liénard equation presents the structure of a dissipative system and as stated above one may investigate the use of the metriplectic formalism to algebraize the dynamics of the equation. Metriplectic systems were introduced by P.J. Morrison [22, 23] and they combine both conservative and dissipative systems. The dynamics of an isolated system with dissipation is regarded as the sum of a Hamiltonian component, generated by H via a Poisson bracket algebra; plus dissipation terms, produced by a certain quantity, called entropy, S via a new symmetric bracket [2, 3, 10, 12, 17, 23]. We also show how to express the Liénard equation in terms of the complex Poisson Hamiltonian equation. In an interesting paper Rajeev [27] has shown that a large class of dissipative systems can be brought to a canonical form by introducing complex co-ordinates in phase space and a complex-valued Hamiltonian. In this paper we identify a class of dissipative systems which yield complex Hamiltonization.

Result and organization In this paper we derive an old result of an integrability criterion of nonlinear differential equation, namely, the Chiellini integrability condition for the Liénard equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$, using the Jacobi last multiplier. It is well known from the work of many authors [13, 14, 19, 20, 8, 21, 29] about the utility of the Chiellini condition for solving nonlinear ODEs, but it is not known its role for the Hamiltonization of the Liénard equation. In this paper we elucidate the construction of Lagrangian and Hamiltonian systems of the Liénard equation using Jacobi's Last Multiplier. In particular, we obtain the bihamiltonian structure of those Liénard equation which satisfies the Chiellini integrability condition (1.1). By combining the Chiellini condition and Hamiltonization of the Liénard equation we show that the Kukles equation is the only equation in the Liénard family which satisfies both the Chiellini integrability and the Sabatini criterion for isochronicity conditions. Using metriplectic dynamics we express the Liénard equation in complex Hamiltonian form with a holomorphic Poisson structure.

This paper is **organized** as follows. Using Jacobi's last multiplier technique and imposing Cheillini integrability condition we formulate the Lagrangian and Hamiltonian description of the Liénard equation in Section 2. In Section 3 we study isochronous Hamiltonian systems connected to Cheillini integrability condition. Section 4 is devoted to metriplectic structure the Liénard equation. We give a complex Hamiltonian formulation of the Liénard equation in Section 5. We end this paper with a modest outlook.

2 The Hamiltonian formulation of a Liénard equation and Cheillini's integrability condition

Consider a class of second order differential equations (ODE) in which the damping term is proportional to the velocity \dot{x} , i.e.,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \quad (2.1)$$

This is equivalent to the standard system,

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x). \quad (2.2)$$

To deduce a Lagrangian for such a planar system we use the Jacobi Last Multiplier (JLM), M , whose relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation of the form

$$\ddot{x} = F(t, x, \dot{x}) \quad (2.3)$$

is

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \quad (2.4)$$

Note that the JLM, $M = M(t, x, \dot{x})$ satisfies, by definition, the following equation

$$\frac{d}{dt}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0, \quad (2.5)$$

which in the present case is

$$\frac{d}{dt}(\log M) - f(x) = 0. \quad (2.6)$$

Assuming that its formal solution is related to a new variable u defined *via*

$$M(t, x) = \exp \left(\int f(x) dt \right) := u^{1/\alpha}$$

we find that

$$\dot{u} = \alpha u f(x). \quad (2.7)$$

Let us now set

$$\dot{x} = u + W(x),$$

where the particular form of $W(x)$ is to be determined. On taking the time derivative of the last equation and using (2.7) we have

$$\ddot{x} = (\alpha f(x) + W'(x))\dot{x} - \alpha f(x)W. \quad (2.8)$$

Comparing (2.1) and (2.8) we see that

$$W'(x) = -(\alpha + 1)f(x) \quad \text{and} \quad \alpha W(x) = \frac{g(x)}{f(x)}.$$

The consistency of these expressions leads to the integrability condition:

$$\frac{d}{dx} \left(\frac{g}{f} \right) = -\alpha(\alpha + 1)f(x), \quad (2.9)$$

which serves to determine the parameter α for given f and g . Comparison of (1.1) and (2.9) shows that the constant $s = -\alpha(1 + \alpha)$. Furthermore the system (2.2) is equivalent to the system

$$\dot{x} = u + \frac{1}{\alpha} \frac{g}{f}, \quad \dot{u} = \alpha u f$$

subject of course to the condition (2.9).

As already mentioned this integrability condition has been employed for obtaining exact solutions of second-order differential equations that can be reduced to an Abel equation of the first kind. A brief outline of the procedure is given below.

2.1 Construction of an exact solution

The third degree polynomial, first kind first-order Abel differential equation is given by

$$\frac{dv}{dx} = f(x)v^2 + g(x)v^3, \quad (2.10)$$

where the coefficients $f(x)$ and $g(x)$ are real valued functions of the variable x . The Liénard system (2.2) can easily be reduced to a first-order equation

$$y \frac{dy}{dx} = -f(x)y - g(x). \quad (2.11)$$

By the substitution $y = 1/v$, (2.11) can be transformed into a first kind Abel differential equation of the form (2.10). Let $z = vg/f$, then by using the Chiellini condition stated in (2.9) one obtains

$$\frac{dz}{dx} = \frac{g}{f} \frac{dy}{dx} + \alpha f y = \frac{f^2}{g} (z^3 + z^2 - \alpha(\alpha + 1)z). \quad (2.12)$$

This is separable and the solution is given by

$$-\frac{1}{\alpha(\alpha + 1)} \ln \left| \frac{g}{f} \right| = \int \frac{dz}{z(z^2 + z + \alpha)} + c, \quad (2.13)$$

where we have used Chiellini's separability condition. This yields implicitly the solution of the Liénard equation.

From the Chiellini condition (2.9) we have

$$g = f(\beta - \alpha(\alpha + 1)) \int f(x) dx = K(x)f, \quad \text{where} \quad K(x) = (\beta - \alpha(\alpha + 1)) \int f(x) dx. \quad (2.14)$$

Thus it is clear that Chiellini condition implies that g and f are not independent and as such the general Liénard equation and the corresponding Abel equation reduce to

$$\ddot{x} + f(x)\dot{x} + K(x)f(x) = 0, \quad \frac{dv}{dx} = f(x)v^2(1 + Kv) \quad (2.15)$$

respectively. It is seen that the restoring force $g(x)$ is proportional to the frictional force $f(x)$, where the proportionality factor $K(x)$ is the primitive (potential) of $f(x)$. If we express (2.15) in terms of a potential function, then it becomes $\ddot{x} + f(x)\dot{x} + KK' = 0$. This yields

$$\frac{1}{2} \frac{d}{dt} (\dot{x}^2 + K(x)^2) = -f(x)\dot{x}^2,$$

which reflects for $f(x) \geq 0$, that the system is stable for all $x \neq 0$. In the following we consider the Hamiltonian formulation of this particular form of the Liénard equation.

2.2 Chiellini integrability and Bi-Hamiltonian structure of polynomial class of Liénard equation

From the above discussion it is clear that the system (2.2) is equivalent to the following system

$$\dot{u} = \alpha u f(x), \quad \dot{x} = u + \frac{1}{\alpha} \left(\frac{g}{f} \right), \quad (2.16)$$

subject to Cheillini's condition for integrability (2.9). Assuming that this system admits a Hamiltonian structure we may recast (2.16) as

$$\dot{x} = u + \frac{1}{\alpha} \frac{g}{f} = -J \frac{\partial H}{\partial u}, \quad \dot{u} = \alpha u f(x) = J \frac{\partial H}{\partial x}, \quad (2.17)$$

where H is the Hamiltonian of the system and J is symplectic, then up on equating the mixed derivative of H w.r.t. x and u we get the following linear partial differential equation determining the symplectic J :

$$J_x \left(u + \frac{1}{\alpha} \frac{g}{f} \right) + \alpha f(x) u J_u = -J f(x). \quad (2.18)$$

The Lagrange system for (2.18) is in general

$$\frac{dx}{u + \alpha^{-1} g/f} = \frac{du}{\alpha u f} = \frac{dJ}{-J f}. \quad (2.19)$$

Its characteristics are easily found to be:

$$c_1 = J u^{1/\alpha} \quad (2.20)$$

$$c_2 = u^{(\alpha+1)/\alpha} \left[\frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right], \quad (2.21)$$

where J is the component of a symplectic matrix. The general solution of (2.19) is therefore of the form $c_1 = F(c_2)$ where F is an arbitrary function. Assuming $F(c_2) = c_2$ we have

$$J = u \left(\frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right). \quad (2.22)$$

It remains to calculate the Hamiltonian H which using the last expression for J in (2.17) is given by

$$H = \ln \left[|u|^{-1/\alpha} \left| \frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right|^{-1/(\alpha+1)} \right]. \quad (2.23)$$

2.3 A Lagrangian and Hamiltonians of Liénard equation

Now from (2.4) and (2.16) we have

$$\frac{\partial^2 L}{\partial \dot{x}^2} = \left(\dot{x} - \frac{1}{\alpha} \frac{g}{f} \right)^{1/\alpha},$$

so that

$$L(x, \dot{x}, t) = \frac{\left(\dot{x} - \frac{1}{\alpha} \frac{g}{f} \right)^{1/\alpha+2}}{(1/\alpha+1)(1/\alpha+2)} + h_1(x, t)\dot{x} + h_2(x, t).$$

Here $h_1(x, t)$ and $h_2(x, t)$ are arbitrary functions of integration. Inserting this Lagrangian into the Euler-Lagrange equation and using (2.1) we find that

$$h_{1t} - h_{2x} = 0.$$

Therefore choosing $h_1(x, t) = G_x$ and $h_2(x, t) = G_t$ it follows that

$$L = \frac{\left(\dot{x} - \frac{1}{\alpha} \frac{g}{f} \right)^{1/\alpha+2}}{(1/\alpha+1)(1/\alpha+2)} + \frac{dG}{dt}. \quad (2.24)$$

We can drop the total derivative term without loss of generality. The conjugate momentum is then defined in the usual manner by

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\left(\dot{x} - \frac{1}{\alpha} \frac{g}{f} \right)^{1/\alpha+1}}{(1/\alpha+1)},$$

and therefore

$$\dot{x} = \frac{1}{\alpha} \frac{g}{f} + ((1/\alpha+1)p)^{1/(1/\alpha+1)}.$$

Using the standard Legendre transformation the Hamiltonian is found to be

$$H = p\dot{x} - L = \frac{1}{\alpha} p \frac{g}{f} + \frac{\alpha}{2\alpha+1} \left(\frac{\alpha+1}{\alpha} p \right)^{\frac{2\alpha+1}{\alpha+1}}. \quad (2.25)$$

The corresponding canonical equations are:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{1}{\alpha} \frac{g}{f} + \left(\frac{\alpha+1}{\alpha} p \right)^{\frac{\alpha}{\alpha+1}}, \quad (2.26)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = (\alpha+1)fp. \quad (2.27)$$

In terms of the scaled variable $\tilde{p} := (\alpha+1)/\alpha p$ the Hamiltonian (2.25) has the following appearance,

$$H(x, \tilde{p}; \alpha) = \frac{1}{\alpha+1} \tilde{p} \frac{g}{f} + \frac{\alpha}{2\alpha+1} \tilde{p}^{\frac{2\alpha+1}{\alpha+1}}. \quad (2.28)$$

Note that α is a parameter and H changes if α changes. The canonical Poisson bracket accordingly becomes $\{x, \tilde{p}\} = (\alpha + 1)/\alpha$. From (2.14) the function $g = K(x)f$ and hence the Hamiltonian assumes the form

$$H(x, \tilde{p}; \alpha) = \frac{\tilde{p}}{\alpha + 1} K(x) + \frac{\alpha}{2\alpha + 1} \tilde{p}^{\frac{2\alpha+1}{\alpha+1}}.$$

3 Isochronicity and the Chiellini integrability condition

In this section we first identify the isochronous cases resulting from the Liénard equation and examine the corresponding Hamiltonian structures. It is shown by Sabatini that if the functions f and g be analytic, g odd, $f(0) = g(0) = 0$ and $g'(0) > 0$ then the origin is an isochronous center if and only if f is odd and

$$\tau(x) := \left(\int_0^x s f(s) ds \right)^2 - x^3 (g(x) - g'(0)x) \equiv 0. \quad (3.1)$$

With out loss of generality we assume $g'(0) = 1$. Thus in case of isochronicity the function $g(x)$ has a specific form depending on $f(x)$ as given by (3.1). If in (3.1) we substitute the expression for $g(x)$ resulting from the Cheillini condition, i.e., $g(x) = K(x)f(x)$ with $K(x) = \beta - \alpha(1 + \alpha) \int_0^x f(s) ds$, we obtain

$$x + \frac{1}{x^3} \left(\int_0^x s f(s) ds \right)^2 = g(x) = \left(\beta - \alpha(\alpha + 1) \int_0^x f(s) ds \right) f(x). \quad (3.2)$$

Assuming $f(x) = kx^\mu$ its substitution into (3.2) gives

$$x + \frac{k^2}{(\mu + 2)^2} x^{2\mu+1} = -\alpha(\alpha + 1) \frac{k^2}{\mu + 1} x^{2\mu+1} + \beta k x^\mu.$$

Choosing $\beta = 1/k$, we are therefore led to conclude that $\mu = 1$. Consequently it follows that the parameter α has to satisfy the following equation, $9\alpha^2 + 9\alpha + 2 = 0$, and its solutions are $\alpha = -1/3$ and $\alpha = -2/3$ respectively.

Hence the Liénard equation admits isochronous motion only when $f(x) = kx$ and $g(x) = x + k^2 x^3/9$. Furthermore with the above values of the parameter α we see that the Hamiltonians, from (2.28), are as follows:

$$H_1(x, \tilde{p}, \alpha = -1/3) = \frac{3}{2} \frac{g}{f} \tilde{p} - \tilde{p}^{1/2}, \quad \text{where} \quad \tilde{p} = -2p,$$

$$H_2(x, \tilde{p}, \alpha = -2/3) = \frac{2}{\tilde{p}} + 3 \frac{g}{f} \tilde{p}, \quad \text{where} \quad \tilde{p} = -\frac{1}{2}p,$$

where $g/f = 1/k + kx^2/9$. Both the Hamiltonians give rise to the same equation, namely

$$\ddot{x} + kx\dot{x} + \left(x + \frac{k^2}{9} x^3 \right) = 0, \quad (3.3)$$

which reduces to the standard Kukles system for the choice $k = 3$. The Poisson brackets corresponding Hamiltonian structures H_1 and H_2 are given by $\{x, \tilde{p}\} = -2$ and $\{x, \tilde{p}\} = -\frac{1}{2}$ respectively. Thus it is seen that the isochronous equation (3.3) admits a bi-Hamiltonian structure.

Thus we have shown that for the Liénard type equations, when the coefficient of the damping term is linear in x , then there is only one equation having an isochronous center, namely (3.3).

3.1 Mapping of a Kukles system to the harmonic/isotonic oscillator equation and isochronicity

In this section we examine the relationship between (3.3) and the equation of a linear harmonic and the isotonic oscillator. These are the only two systems with a rational potential which display the property of isochronicity. The system (3.3) is equivalent to

$$\dot{u} = \alpha u k x, \quad \dot{x} = u + \frac{1}{\alpha k} + \frac{k}{9\alpha} x^2, \quad (3.4)$$

with $9\alpha^2 + 9\alpha + 2 = 0$. The associated symplectic structure may be taken as $J = cu^{-1/\alpha}$ for the allowed values of $\alpha = -1/3$ and $-2/3$. The Hamiltonian for (3.4) with this symplectic form turns out to be

$$H = \frac{\alpha k}{2c} x^2 u^{(\alpha+1)/\alpha} + V(u) + s, \quad (3.5)$$

where s is a constant and the potential

$$V(u, \alpha) = -\frac{1}{ck(\alpha+1)} u^{(\alpha+1)/\alpha} - \frac{\alpha}{c(2\alpha+1)} u^{(2\alpha+1)/\alpha}. \quad (3.6)$$

Case 1: When $\alpha = -1/3$ the expression for the potential simplifies to

$$V(u, \alpha = -1/3) = -\frac{3}{2ck} \left(\frac{1}{u} - \frac{k}{3} \right)^2 + \frac{k}{6c}.$$

By defining a new set of variables $\xi := \frac{x}{u}$ and $\eta := \frac{1}{u} - \frac{k}{3}$ the Hamiltonian may be expressed as

$$H_{h.o} = -\frac{k}{6c} \xi^2 - \frac{3}{2ck} \eta^2$$

after setting $s = -k/6c$. The overall negative sign can be fixed by an appropriate choice of the values of k and c . For instance if $k = 3$ and $c = -1$ then the above Hamiltonian reduces to the standard form

$$H_{h.o} = \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2, \quad \{\xi, \eta\} = -1. \quad (3.7)$$

Case 2: For $\alpha = -2/3$ the choice $\xi := \frac{x}{u^{1/4}}$, $\eta := \frac{2}{u^{1/4}}$ causes the Hamiltonian to become

$$H_{iso} = -\frac{k}{3c} \xi^2 - \frac{2}{c} \frac{4}{\eta^2} - \frac{3}{ck} \frac{\eta^2}{4},$$

which upon choosing $k = 3, c = -2$ reduces to the form

$$H_{iso} = \frac{1}{2}\xi^2 + \frac{1}{8}\eta^2 + \frac{4}{\eta^2}, \quad \{\xi, \eta\} = -1, \quad (3.8)$$

and corresponds to that of the isotonic oscillator. The latter is also known as the singular harmonic oscillator as the singular term has the effect of centrifugal type of potential. The classical motion is confined either to the region $\eta > 0$ or $\eta < 0$ and the corresponding quantum system is Schrödinger solvable with an equispaced spectrum.

3.1.1 Solution via Chiellini integrability condition

We complete this article by obtaining the solution of the Kukles equation using Chiellini integrability condition. Once again consider the Liénard equation in the form

$$\ddot{x} + \lambda x \dot{x} + \beta x^3 + \gamma x = 0 \quad (3.9)$$

Here $f(x) = \lambda x$ and $g(x) = \gamma x + \beta x^3$ and the Chiellini condition is given by

$$\frac{d}{dx} \left(\frac{g(x)}{f(x)} \right) = \frac{2\beta}{\lambda^2} f(x).$$

Set $\dot{x} = c_k \frac{g(x)}{f(x)}$, where c_k is a constant to be determined, and differentiate this once more to obtain

$$\ddot{x} = c_k^2 \frac{2\beta}{\lambda^2} g(x),$$

where we have used the Chiellini condition. Substituting the first and second-order derivatives of x into (3.9) we have

$$c_k^2 \frac{2\beta}{\lambda^2} g(x) + f(x) c_k \frac{g(x)}{f(x)} + g(x) = 0,$$

which immediately determines the constant c_k as

$$c_k = \frac{-\lambda^2 \pm \sqrt{\lambda^2 - 8\beta^2}}{4\beta}. \quad (3.10)$$

As $g(x) = \beta x^3 + \gamma x$ and $f(x) = \lambda x$, it follows from $\dot{x} = c_k \frac{g(x)}{f(x)}$ that

$$\frac{1}{\sqrt{\frac{\gamma}{\beta}}} \tan^{-1} \frac{x}{\sqrt{\frac{\gamma}{\beta}}} = \frac{\beta}{\lambda} c_k (t - t_0). \quad (3.11)$$

4 Chiellini integrability condition and metriplectic structure

In this section we show that using Chiellini integrability condition the Liénard equation can be reformulated in terms of the complex Hamiltonian theory. To this end we define

$$V_x = \frac{g}{f}, \quad V_{xx} \equiv \frac{d}{dx}\left(\frac{g}{f}\right) = \mu f. \quad (4.1)$$

Therefore in terms of the new function V a Liénard type ODE has the form therefore be written as

$$\ddot{x} + \frac{1}{\mu} V_{xx} \dot{x} + \left(\frac{1}{2\mu} V_x^2\right)_x = 0. \quad (4.2)$$

As a first-order system of ODEs it may be recast as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{1}{\mu} V_{xx} y - \left(\frac{1}{2\mu} V_x^2\right)_x. \end{aligned} \quad (4.3)$$

Lemma 4.1 *The Liénard equation transforms under*

$$y = p - V_x. \quad (4.4)$$

to new set of first order ODE

$$\begin{aligned} \dot{x} &= p - V_x, \\ \dot{p} &= -\left(\frac{1}{\mu} - 1\right) V_{xx} p - \left(\frac{1}{2\mu} V_x^2\right)_x. \end{aligned} \quad (4.5)$$

Proof: It is clear that

$$\begin{aligned} \dot{p} &= V_{xx}(p - V_x) - \frac{1}{\mu} V_{xx}(p - V_x) - \frac{1}{2\mu} (V_x^2)_x \\ &= -\left(\frac{1}{\mu} - 1\right) V_{xx} p - \left(\frac{1}{2\mu} V_x^2\right)_x. \quad \square \end{aligned}$$

Our aim is next to rewrite the system (4.3) in a metriplectic and complex form. Let S be a real valued function on a m -dimensional manifold M . If M is compact smooth Riemannian manifold, the gradient vector field associated with the metric $g = \sum g_{ij} dx^i \otimes dx^j$ is given by

$$\text{grad}(S) = G\left(\frac{\partial S}{\partial x_1}, \dots, \frac{\partial S}{\partial x_m}\right),$$

where $G = (g_{ij})$ and (x_1, \dots, x_m) is a local coordinate.

P.J. Morrison [22] introduced a natural geometrical formulation of dynamical systems that exhibit both conservative and nonconservative characteristics. A metriplectic system [2, 3, 10, 12, 17, 22] consists of a smooth manifold M , two smooth vector bundle maps $J^\sharp, G^\sharp : T^*M \rightarrow TM$ covering the identity, and two functions $H, S \in C^\infty(M)$, the Hamiltonian or total energy and the entropy of the system, such that it yields Poisson bracket and positive semidefinite symmetric bracket

$$J(df, dh) = \{f, h\}, \quad G(df, dh) = (f, g),$$

respectively. Moreover, the additional requirements that H remains a conserved quantity and S continues to be dissipated. These requirements can be met if the following conditions on H and S are satisfied $\{S, F\} = 0$ and $(H, F) = 0$ for all $F \in C^\infty(M)$, i.e, $JdS = GdH = 0$. It shows that S is a Casimir function for the Poisson tensor J and dH is a null vector for the symmetric tensor G .

In this paper we work with a slightly weaker condition of metriplectic condition, i.e. $JdS + GdH = 0$.

Proposition 4.1 *The Liénard equation of motion take the following form*

$$\dot{X} = J\nabla H_1 - G\nabla S, \tag{4.6}$$

where $X = \begin{pmatrix} x \\ p \end{pmatrix}$. Here J is the standard symplectic matrix and the second term represents gradient flow, where G is defined by

$$G = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \alpha \end{pmatrix},$$

where α is a parameter. The H_1 and S are given by

$$H_1 = \frac{1}{2}p^2 + \frac{1}{2\mu}V_x^2 + \left(\left(\frac{1}{\mu} - 1\right)V_x - \alpha x\right)p, \tag{4.7}$$

$$S = \frac{1}{2}p^2 + \alpha\left(\frac{1}{\mu}V - \frac{\alpha}{2}x^2\right). \tag{4.8}$$

Proof: It is easy to see that

$$H_{1x} = \left[\frac{1}{\mu}V_x V_{xx} + \left(\left(\frac{1}{\mu} - 1\right)V_{xx} - \alpha\right)p\right], \tag{4.9}$$

$$H_{1p} = p + \left(\left(\frac{1}{\mu} - 1\right)V_x - \alpha x\right), \tag{4.10}$$

$$S_x = \alpha\left[\frac{1}{\mu}V_x - \alpha x\right], \tag{4.11}$$

and $S_p = p$. Using all these expressions we obtain our result. \square

Corollary 4.1 *If $\mu = 2$ and $p = V_x$ then the Liénard equation satisfies weaker metriplectic condition, i.e., $JdS + GdH = 0$.*

5 Complex Hamiltonian formulation and Liénard equation

Suppose S be the symplectic foliation of M . We denote by N the distribution defined as the g -orthogonal complement to S . Thus at every point m a decomposition into direct sum of sub-bundles, i.e. $T_m M = T_m S \oplus N_x$. If the Poisson bivector Π is parallel with respect to the Levi-Civita connection ∇ , i.e. $\nabla \Pi = 0$. There is a classical result of Lichnerowicz ([18],[32], Remark 3.11)) that the distribution N is integrable. Hence together with the symplectic structure and the restriction of the metric g to the symplectic leaves defines a Kähler structure.

It is also possible to express the Hamiltonian of an equation of the Liénard type within the framework of the complex Hamiltonian theory. In this section we adopt more straight forward approach, and once again we demonstrate the role of Chiellini integrability condition.

Proposition 5.1 *The equations of motion take the complex form, given by*

$$\frac{d}{dt} \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p\} \\ \{H_1, x\} \end{pmatrix} + J \begin{pmatrix} \{H_2, p\} \\ \{H_2, x\} \end{pmatrix}, \quad (5.1)$$

where J is an almost complex structure defined by

$$J = \begin{pmatrix} 0 & -\alpha \\ \frac{1}{\alpha} & 0 \end{pmatrix},$$

where α is a parameter. The Hamiltonians are given by

$$H_1 = \frac{1}{2}p^2 + \frac{1}{2\mu}V_x^2 + \left(\left(\frac{1}{\mu} - 1\right)V_x - \alpha x\right)p, \quad (5.2)$$

$$H_2 = \frac{1}{2}p^2 + \alpha\left(\frac{1}{\mu}V - \frac{\alpha}{2}x^2\right). \quad (5.3)$$

A complex structure allows one to endow a real vector space \mathcal{V} with the structure of a complex vector space. In other words, given any real vector space \mathcal{V} we may define its complexification by $\mathcal{V}^{\mathbb{C}} = \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ and J is guaranteed to have eigenvalues which satisfy $a^2 = -1$, namely $a = \pm i$. Thus we may write

$$\mathcal{V}^{\mathbb{C}} = \mathcal{V}^+ \oplus \mathcal{V}^-$$

where \mathcal{V}^+ and \mathcal{V}^- are the eigenspaces of $+i$ and $-i$ respectively. Given such a matrix J we can define the equation of motion in terms of the complex coordinates

$$\dot{z} = \{H_{\mathbb{C}}, z\}, \quad (5.4)$$

generated by the complex Hamiltonian function $H_{\mathbb{C}} = H_1 + iH_2$, where H_1 and H_2 are as given in (5.2) and (5.3) respectively.

Lemma 5.1 *The Liénard equation can be recast as*

$$\dot{z} = \{H_1 + iH_2, z\}, \quad (5.5)$$

where $z = \frac{1}{\sqrt{2\alpha}}(\alpha x - ip)$. The conjugate $z^* = \frac{1}{\sqrt{2\alpha}}(\alpha x + ip)$ and z satisfy

$$\{z^*, z\} = -i. \quad (5.6)$$

Proof: Equating real and imaginary part we obtain

$$\alpha \dot{x} = \alpha \{H_1, x\} + \{H_2, p\}, \quad -\dot{p} = \alpha \{H_2, x\} - \{H_1, p\}.$$

By normalizing these set of we obtain that these equations are equivalent to \dot{p} and \dot{x} equations. Thus we find our desired result. Moreover it is easy to check directly that $\{z^*, z\} = -i$. \square

By changing coordinates $(x, p) \rightarrow (z, z^*)$ one rewrite the Hamiltonian equation in complex form in terms of the complex Poisson bracket

$$\{K, L\} = -i \left(\frac{\partial K}{\partial z^*} \frac{\partial L}{\partial z} - \frac{\partial K}{\partial z} \frac{\partial L}{\partial z^*} \right). \quad (5.7)$$

Proposition 5.2 *Suppose the Chiellini integrability condition is satisfied for the Liénard equation of motion. Then Liénard equation can be expressed in complex Hamiltonian form*

$$\dot{z} = \{H_{\mathbb{C}}, z\} = -i \frac{\partial H_{\mathbb{C}}}{\partial z^*}, \quad (5.8)$$

with complex coordinates and complex Hamiltonian function $H_{\mathbb{C}} = H_1 + iS$, where H_1 and S are as given in (5.2) and (5.3) respectively.

6 Concluding remarks

In this paper we have studied the classic Chiellini integrability condition $\frac{d}{dx}(\frac{q}{f}) = \alpha(1-\alpha)f(x)$, and its role in the construction of Lagrangian and Hamiltonian of the Liénard equation. We have formulated the Chiellini condition using the Jacobi last multiplier, and show that the only equation which satisfies both the Chiellini condition and Sabatini's isochronicity condition is the Kukles system. We have re-examined the result by mapping the Liénard equation to harmonic oscillator equation using the α values of the Chiellini integrability condition.

The Liénard equation exhibits many interesting feature of dynamics, we only focus on integrability or isochronicity aspects. In other words we study the Lagrangians and Hamiltonians of the those Liénard system which have centers. It would be interesting to study Hamiltonian aspects of the Liénard equation which has attractor and other dynamical features, one such work studied dissipative dynamical systems capable of showing limit cycle oscillations using canonical perturbation theory [31].

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